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RADIATION WITH FREE CONVECTION IN AN ABSORBING, EMITTING AND SCATTERING MEDIUM

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Abstract—Heat transfer by simultaneous radiation and free convection from a vertical plate immersed in an absorbing, emitting, isotropically scattering, gray fluid is determined by solving non-similar momentum and energy equations. The radiation part of the problem is treated exactly by the application of the normal-mode-expansion technique. The hot wall and the cold wall conditions are both considered. A parameter survey is made to study the effects of single scattering albedo, optical thickness, and conduction-to-radiation parameter on temperature distribution in the boundary layer and heat transfer at the wall.

NOMENCLATURE

- c_p, specific heat at constant pressure;
- f, dimensionless stream function;
- I^*, I, radiation intensity (dimensional and dimensionless respectively);
- k, thermal conductivity;
- N, conduction-to-radiation parameter;
- Pr, Prandtl number;
- q, Q, dimensional and dimensionless heat flux respectively;
- T, temperature;
- u, v, velocity components in x- and y-direction respectively;
- x, y, coordinates parallel and perpendicular to the plate respectively;
- β, extinction coefficient;
- ε, emissivity;
- η, dimensionless variable as defined by equation (3a);
- θ, dimensionless temperature;
- θ^*, functions as defined by equations (12a) and (12b);
- ρ, density;
- σ, Stefan–Boltzmann constant;
- τ, optical variable;
- ψ, stream function;
- ω, single scattering albedo.

Subscripts

- c, refers to cold wall;
- h, refers to hot wall;
- w, refers to the wall;
- ∞, refers to outside the thermal boundary layer;

Superscripts

- r, refers to radiation;
- t, refers to total.

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INTRODUCTION

HEAT transfer by simultaneous forced convection and radiation in a participating fluid has received great deal of attention during the past decade, but only limited amount of work is available in the area of interaction of radiation with free convection. Cess [1] considered an absorbing and emitting fluid in the optically thick region and used the singular perturbation technique to solve the problem. Arpaci [2] has investigated a similar problem in both the optically thin and thick regions and used the approximate integral technique and a first order profile to solve the energy equation. The accuracy of approximate solutions can not be predicted unless they are compared with the exact results. The purpose of the present work is to present highly accurate results on the interaction of radiation with free convection including the effects of scattering. The radiation part of the problem is treated exactly by the application of the normal-mode-expansion technique developed originally by Case [3] and an iterative numerical scheme is used to solve the resulting system of equations. Effects of scattering albedo, optical thickness, the conduction-to-radiation parameter are investigated on the temperature profile in the boundary layer and heat transfer at the wall. The results of the present analysis can be used to check the accuracy of solutions obtained with approximate methods.

ANALYSIS

Consider simultaneous radiation and free-convection from a vertical plate at a uniform temperature T_w immersed in an absorbing, emitting, isotropically scattering gray fluid of infinite extend at temperature T_∞ as illustrated in Fig. 1. Assuming that the difference between

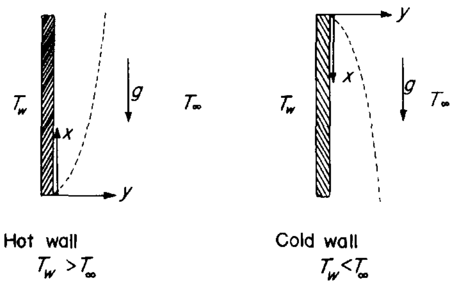

FIG. 1. Geometry and coordinate system.

the plate and fluid temperature is sufficiently small, the governing continuity, momentum and energy equations are given as:

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1a}$$

momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \bar{v}\frac{\partial^2 u}{\partial y^2} \pm g\lambda(T - T_\infty) \tag{1b}$$

energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q^r}{\partial y} \tag{1c}$$

where the positive sign in the last term of the momentum equation characterizes the hot wall condition ($T_w > T_\infty$) and the negative sign the cold wall condition ($T_w < T_\infty$). Here λ is the coefficient of thermal expansion, g is the acceleration of gravity, q^r is the net radiative heat flux in the y-direction, α is the thermal diffusivity, and $\bar{v}$ is the kinematic viscosity. We note that this set of equations is similar to the conventional equations for laminar free convection from a vertical plate except for the radiation term in the energy equation.

The boundary conditions are taken as

$$u = v = 0,\ T = T_w \text{ at } y = 0 \tag{2a}$$

$$u \to 0, \qquad T \to T_\infty \text{ at } y \to \infty. \tag{2b}$$

The above equations are now transformed by defining the following new variables:

$$\eta \equiv \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x} \tag{3a}$$

$$Gr_x \equiv \frac{g\lambda|T_w - T_\infty|x^3}{\bar{v}^2} \quad \text{(local Grashof number)} \tag{3b}$$

$$\psi(x,\eta) \equiv 4\bar{v}\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} f(x,\eta) \tag{3c}$$

$$\theta(x,\eta) \equiv \frac{T}{T_w} \quad \text{(for hot wall)} \tag{4a}$$

$$\equiv \frac{T}{T_\infty} \quad \text{(for cold wall)} \tag{4b}$$

$$Q^r \equiv \frac{q^r}{4\sigma T_w^4} \quad \text{(for hot wall)} \quad (5a)$$

$$\equiv \frac{q^r}{4\sigma T_\infty^4} \quad \text{(for cold wall)} \quad (5b)$$

$$N \equiv \frac{k\beta}{4\sigma T_w^3} \quad \text{(for hot wall)} \quad (6a)$$

$$\equiv \frac{k\beta}{4\sigma T_\infty^3} \quad \text{(for cold wall).} \quad (6b)$$

Here the stream function ψ is related to the velocity components u and v by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \quad (7)$$

Then continuity equation is identically satisfied. The momentum and the energy equations are transformed respectively to:

momentum

$$\frac{\partial^3 f}{\partial \eta^3} + \theta^* - 2\left(\frac{\partial f}{\partial \eta}\right)^2 + 3f\frac{\partial^2 f}{\partial \eta^2} = 4x\left(\frac{\partial^2 f}{\partial x \partial \eta}\frac{\partial f}{\partial \eta} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial \eta^2}\right) \quad (8a)$$

energy

$$\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + 3f\frac{\partial \theta}{\partial \eta} = 4x\left(\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial \eta} + \frac{\beta x}{NPr\,(Gr_x/4)^{\frac{1}{4}}}\frac{\partial Q^r}{\partial \eta}\right). \quad (8b)$$

A new independent variable ξ (Bouguer number), which characterizes the optical thickness, is defined as

$$\xi \equiv \frac{\beta x}{(Gr_x/4)^{\frac{1}{4}}}. \quad (9)$$

Then the momentum and the energy equations can be written in the alternative form as

$$\frac{\partial^3 f}{\partial \eta^3} + \theta^* - 2\left(\frac{\partial f}{\partial \eta}\right)^2 + 3f\frac{\partial^2 f}{\partial \eta^2} = \xi\left(\frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial f}{\partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) \quad (10)$$

and

$$\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + 3f\frac{\partial \theta}{\partial \eta} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial \theta}{\partial \eta} + \frac{1}{NPr}\frac{\partial Q^r}{\partial \eta}\right) \quad (11)$$

where we have defined

$$\theta^* = \theta_h^* \equiv \frac{\theta - \theta_\infty}{1 - \theta_\infty} \quad \text{(for hot wall)} \quad (12a)$$

$$= \theta_c^* \equiv \frac{1 - \theta}{1 - \theta_w} \quad \text{(for cold wall)} \quad (12b)$$

where

$$\theta_\infty = \frac{T_\infty}{T_w}$$

$$\theta_w = \frac{T_w}{T_\infty}.$$

The boundary conditions, equations (2a) and (2b), are transformed to:

for the hot wall

$$f = \frac{\partial f}{\partial \eta} = 0, \theta = 1 \text{ at } \eta = 0 \quad (13a)$$

$$\frac{\partial f}{\partial \eta} = 0, \theta = \theta_\infty \text{ at } \eta \to \infty \quad (13b)$$

and for the cold wall

$$f = \frac{\partial f}{\partial \eta} = 0, \theta = \theta_w \text{ at } \eta = 0 \quad (14a)$$

$$\frac{\partial f}{\partial \eta} = 0, \theta = 1 \quad \text{at } \eta \to \infty. \quad (14b)$$

The transformed momentum and energy equations (10) and (11) involve partial derivatives with respect to the variable ξ, additional boundary conditions are needed. They are chosen as

$$\theta = \theta_0 \text{ and } f = f_0 \text{ for } \xi = 0 \quad (15)$$

where θ_0 and f_0 represent the solution of the considered problem for the non-radiative case.

The energy equation (11) involves the radiative heat flux Q^r which should be obtained from the solution of the equation of radiative heat transfer subject to appropriate boundary conditions.

RELATION FOR THE RADIATION HEAT FLUX

We consider the fluid as an absorbing, emitting, isotropically scattering, semi-infinite $(0 \leqslant \tau < \infty)$ region stratified in planes perpendicular to the τ-axis. The equation of radiative transfer for the dimensionless radiation intensity $I(\xi, \tau, \mu)$ is given as

$$\frac{\partial I(\xi,\tau,\mu)}{\partial \tau} + I(\xi,\tau,\mu) = (1-\omega)\,\theta^4(\xi,\tau) + \frac{\omega}{2} \int_{-1}^{+1} I(\xi,\tau,\mu')\,d\mu' \text{ in } 0 \leqslant \tau < \infty,\ -1 \leqslant \mu \leqslant 1. \tag{16}$$

For a diffusely emitting, specularly reflecting, opaque plate the boundary condition at $\tau = 0$ is given as

$$I(\xi, 0, \mu) = \varepsilon_w \theta_r^4 + (1-\varepsilon_w)\, I(\xi, 0, -\mu),\ \mu > 0 \tag{17}$$

where we have assumed the Kirchhoff law is valid. The case $\varepsilon_w = 1$ characterize a black boundary. The dimensionless intensity $I(\xi, \tau, \mu)$ is related to the radiation intensity $I^*(\xi, \tau, \mu)$ by

$$I(\xi,\tau,\mu) = \frac{I^*(\xi,\tau,\mu)}{\sigma T_w^4/\pi} \quad \text{(for hot wall)} \tag{18a}$$

$$= \frac{I^*(\xi,\tau,\mu)}{\sigma T_\infty^4/\pi} \quad \text{(for cold wall)} \tag{18b}$$

and θ_r is the dimensionless temperature at the wall defined as

$$\theta_r = \frac{T_w}{T_w} = 1 \qquad \text{(for hot wall)}$$

$$\theta_r = \frac{T_w}{T_\infty} = \theta_w \qquad \text{(for cold wall)}$$

and at $\tau \to \infty$ the solution $I(\xi, \tau, \mu)$ should remain finite. Here τ is the optical variable defined as

$$d\tau = \beta\, dy. \tag{18c}$$

ω is the single scattering albedo, μ is the cosine of the angle between the direction of radiation intensity and the positive τ-axis and $\theta(\xi, \tau)$ is the dimensionless temperature distribution in the fluid region $(0 \leqslant \tau < \infty)$ defined as

$$\theta(\xi,\tau) = \frac{T(\xi,\tau)}{T_w} \quad \text{(for hot wall)} \tag{18d}$$

$$\theta(\xi,\tau) = \frac{T(\xi,\tau)}{T_\infty} \quad \text{(for cold wall).} \tag{18e}$$

Here we note that in equation (16) the radiation intensity $I(\xi, \tau, \mu)$ depends on the ξ variable because the temperature $\theta(\xi, \tau)$ is function of both ξ and τ. Therefore, ξ enters equation (16) merely as a parameter. Furthermore, the equation of radiative transfer is given in the optical variable τ whereas the energy equation involves the independent variable η which is related to τ by

$$\eta \equiv \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x} \equiv \frac{\beta y}{\xi} = \frac{\tau}{\xi}. \tag{19}$$

The dimensionless radiative heat flux is defined as

$$Q^r(\xi,\tau) \equiv \tfrac{1}{2} \int_{-1}^{+1} I(\xi,\tau,\mu')\,\mu'\,d\mu'. \tag{20}$$

The normal-mode-expansion technique is used to solve the equation of radiative transfer subject to the above boundary conditions. The solution of equation (16) is written as a linear sum of the homogeneous solution of this equation and a particular solution $I_p(\xi, \tau, \mu)$ in the form [5, 6]

$$I(\xi,\tau,\mu) = A(v_0,\xi)\,e^{-\tau/v_0}\,\phi(v_0,\mu) + \int_0^1 A(v,\xi)\,e^{-\tau/v}\,\phi(v,\mu)\,dv + I_p(\xi,\tau,\mu) \tag{21}$$

where we have omitted those homogeneous solutions that diverge at infinity, hence this solution satisfies the boundary condition at infinity. Here the discrete normal-mode $\phi(v_0, \mu)$

and the continuum normal-mode $\phi(\nu, \mu)$ are defined as

$$\phi(\nu_0, \mu) = \frac{\omega\nu_0}{2}\frac{1}{\nu_0 - \mu}, \; \nu_0 \notin (-1, 1) \quad (22a)$$

$$\phi(\nu, \mu) = \frac{\omega\nu}{2}\frac{P}{\nu - \mu}$$

$$+ (1 - \omega\nu \tanh^{-1}\nu)\,\delta(\nu - \mu), \; \nu \in (-1, 1) \quad (22b)$$

and the discrete eigenvalues ν_0 are the zeros of the dispersion function

$$\Lambda(\nu_0) \equiv 1 - \omega\nu_0 \tanh^{-1}\frac{1}{\nu_0} \quad (22c)$$

P is a mnemonic symbol used to denote the Cauchy principal value integral and $\delta(x)$ is the Dirac delta function. The readers are invited to refer to [5] for a detailed discussion of the properties of the normal modes. We note that the solution equation (21) satisfies both the integral equation (16) and the requirement of the boundary condition at $\tau \to \infty$. Then the two unknown expansion coefficients $A(\nu_0, \xi)$ and $A(\nu, \xi)$ can be determined by constraining the solution equation (21) to satisfy the boundary condition equation (17) and by utilizing the orthogonality property of the normal modes and various normalization integrals as discussed in [5] and [6], provided that a particular solution $I_p(\xi, \tau, \mu)$ of the equation of radiative transfer is available; but a particular solution can not be determined until the temperature distribution $\theta(\xi, \tau)$ in the fluid is known. At this point, we assume an initial guess for the temperature distribution $\theta(\xi, \tau)$ and that the fourth power of this temperature is represented in the form:

$$\theta^4(\xi, \tau) = \theta_\infty^4 + S_h(\xi, \tau) \quad \text{(for hot wall)} \quad (23a)$$

$$\theta^4(\xi, \tau) = 1 - S_c(\xi, \tau) \quad \text{(for cold wall)} \quad (23b)$$

where the functions S_h and S_c are such that they vanish at $\tau \geqslant \tau_0$ (or $\eta \geqslant \eta_0$), where τ_0 (or η_0) is the edge of the thermal boundary layer; and at the wall, S_h equals $(1 - \theta_\infty^4)$ where $\theta_\infty = T_\infty/T_w$, and S_c equals $(1 - \theta_w^4)$ where $\theta_w = T_w/T_\infty$. Several particular solutions of the equation of radiative transfer are available [4]. Therefore, it is desirable to express the functions S_h and S_c by the superposition of those function for which particular solutions of the equation of radiative transfer are available. We have chosen a truncated cosine series representation of $S(\xi, \tau)$ function in the form

$$S(\xi, \tau) = \sum_{m=0}^{M} B_m(\xi) \cos\frac{m\pi\tau}{\tau_0} \quad (24a)$$

$$= \sum_{m=0}^{M} B_m(\xi) \cos\frac{m\pi\eta}{\eta_0}. \quad (24b)$$

Once the function $S(\xi, \tau)$ is prescribed at any position ξ, the coefficients $B_m(\xi)$ are determined by equation (24) (In the present analysis a 21-term expansion is used to represent the function.) Knowing the coefficients $B_m(\xi)$, particularly solutions of the equation of radiative transfer for an inhomogeneous term in the form: for the hot wall

$$(1 - \omega)\,\theta^4(\xi, \tau) \equiv (1 - \omega)\,[\theta_\infty^4 + S_h(\xi, \tau)] \quad (25a)$$

and for the cold wall

$$(1 - \omega)\,\theta^4(\xi, \tau) \equiv (1 - \omega)\,[1 - S_c(\xi, \tau)] \quad (25b)$$

are obtained respectively as [4]:
for the hot wall

$$I_p(\xi, \eta, \mu) = (1 - \omega)\left\{\frac{\theta_\infty^4}{1 - \omega} + \sum_{m=0}^{M} B_m(\xi) \times \frac{\frac{\xi\eta_0}{m\pi}\left[\frac{\xi\eta_0}{m\pi}\cos\frac{m\pi\eta}{\eta_0} + \mu\sin\frac{m\pi\eta}{\eta_0}\right]}{\left[1 - \frac{\omega\eta_0\xi}{m\pi}\tanh^{-1}\left(\frac{m\pi}{\xi\eta_0}\right)\right]\left[\left(\frac{\xi\eta_0}{m\pi}\right)^2 + \mu^2\right]}\right\} \quad (26a)$$

and for the cold wall

$$I_p(\xi, \eta, \mu) = (1 - \omega)\left\{\frac{1}{1 - \omega} - \sum_{m=0}^{M} B_m(\xi)\right.$$

$$\times \frac{\dfrac{\xi\eta_0}{m\pi}\left[\dfrac{\xi\eta_0}{m\pi}\cos\dfrac{m\pi\eta}{\eta_0}+\mu\sin\dfrac{m\pi\eta}{\eta_0}\right]}{\left[1-\dfrac{\omega\eta_0\xi}{m\pi}\tanh^{-1}\dfrac{m\pi}{\xi\eta_0}\right]\left[\left(\dfrac{\xi\eta_0}{m\pi}\right)^2+\mu^2\right]}\Bigg\}. \tag{26b}$$

Knowing a particular solution $I_p(\xi, \tau, \mu)$ [or $I_p(\xi, \eta, \mu)$] and the expansion coefficients $A(v_0, \xi)$ and $A(v, \xi)$, the dimensionless net radiative heat flux $Q^r(\xi, \eta)$ is obtained from equation (20) and (21) as

$$Q^r(\xi,\eta) = \tfrac{1}{2}(1-\omega)\quad v_0 A(v_0,\xi)\,e^{-\xi\eta/v_0} + \int_0^1 v A(v,\xi)\,e^{-\xi\eta/v}\,dv + \frac{1}{1-\omega}\int_{-1}^{+1} I_p(\xi,\eta,\mu')\,\mu'\,d\mu' \tag{27}$$

where we have transformed the independent variable τ to η by the transformation equation (19). Differentiating equation (27) with respect to η we obtain the desired relation for $\partial Q^r/\partial\eta$ appearing in the energy equation (11) as:

$$\frac{\partial Q^r(\xi,\eta)}{\partial\eta} = \frac{-1}{2}\xi(1-\omega)\,[A(v_0,\xi)\,e^{-\xi\eta/v_0} + \int_0^1 A(v,\xi)\,e^{-\xi\eta/v}\,dv - \frac{1}{\xi(1-\omega)}\int_{-1}^{+1}\frac{\partial}{\partial\eta} I_p(\xi,\eta,\mu')\,\mu'\,d\mu' \tag{28}$$

where the last integral in the bracket is evaluated by equation (26).

Substitution of equations (28) and (29) into equation (11) gives the final form of the energy equation as

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + 3f\frac{\partial\theta}{\partial\eta} = \xi\left(\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right) - \frac{\xi^2}{2PrN}(1-\omega)\times\Bigg[A(v_0,\xi)\,e^{-\xi\eta/v_0} + \int_0^1 A(v,\xi)\,e^{-\xi\eta/v}\,dv \mp 2\sum_{m=0}^{M} B_m(\xi)$$

$$\times\frac{1-\dfrac{\xi\eta_0}{m\pi}\tan^{-1}\dfrac{m\pi}{\xi\eta_0}}{1-\dfrac{\omega\eta_0\xi}{m\pi}\tanh^{-1}\dfrac{m\pi}{\xi\eta_0}}\cos\frac{m\pi\eta}{\eta_0}\Bigg] \tag{29}$$

where the minus sign and plus sign in the last term on the right-hand side refer to the hot wall and cold wall respectively. It is to be noted that when $\xi = 0$, this equation reduces to that of pure free convection.

The total net heat flux at the wall q_w^t is given by

$$q_w^t = \left[-k\frac{\partial T}{\partial y} + q^r\right]_{y=0} \tag{30}$$

which is expressed in terms of the dimensionless quantities as

$$Q_w^t = \frac{q_w^t}{4\sigma T_r^4} = \left[-\frac{N}{\xi}\frac{\partial\theta}{\partial\eta} + Q^r\right]_{\eta=0}, \quad \xi \neq 0 \tag{31}$$

where $T_r = T_w$ for the hot wall, $T_r = T_\infty$ for the cold wall and the dimensionless net radiative heat flux Q^r is given by equation (27). A local Nusselt number, Nu_x is defined as

$$Nu_x \equiv \frac{hx}{k} = \frac{q_w^t x}{k|T_w - T_\infty|} \tag{32}$$

where h is the local heat transfer coefficient. Substituting equation (31a) into equation (32), the local Nusselt number, Nu_x, is given as

$$Nu_x\left(\frac{Gr_x}{4}\right)^{-\frac{1}{4}} = \frac{1}{1-\theta_r}\left[-\frac{\partial\theta}{\partial\eta} + \frac{\xi}{N}Q^r\right]_{\eta=0} \tag{33}$$

where $\theta_r = \theta_\infty$ for the hot wall and $\theta_r = \theta_w$ for the cold wall. For large values of N or for $\xi/N \ll 1$, the radiation term in this equation is negligible and the resulting expression agrees with the standard relation for the Nusselt number given by Ostrach [8].

METHOD OF SOLUTION

To determine the temperature distribution in the medium, the partial differential equations (10) and (29) should be solved simultaneously with appropriate boundary conditions. In the present analysis, these partial differential equations are transformed into ordinary differential equations in the η variable by replacing all the derivatives with respect to ξ by finite differences i.e.

$$\left.\frac{\partial \theta}{\partial \xi}\right|_i = \frac{\theta_i - \theta_{i-1}}{\Delta\xi}, \left.\frac{\partial f}{\partial \xi}\right|_i = \frac{f_i - f_{i-1}}{\Delta\xi}, \text{etc.}$$

The resulting equations are integrated in the η-direction by a Runge–Kutta method at each nodal point ξ_i by starting the calculations at $\xi = 0$. The temperature distribution in the previous station ξ_{i-1} is used as an initial guess to start the calculation at the station ξ_i. The fourth power of this temperature distribution is represented in a 21-term cosine expansion as given by equations (23) and (24) and the corresponding coefficients $B_{m,i}$ and the expansion coefficients $A(\nu_0, \xi_i)$ and $A(\nu, \xi_i)$ are determined. Knowing these coefficients, the momentum and energy equations are integrated with a Runge–Kutta method and a first approximation is obtained for the velocity and temperature profiles. The first approximation for the temperature profile is then used to obtain a second approximation and so forth. This procedure is repeated until the solution has converged to a prescribed criterion, namely, the numerical values of $(\partial\theta/\partial\eta)|_{\eta=0}$ and $(\partial^2 f/\partial\eta^2)|_{\eta=0}$ between the successive iterations differ no more than 10^{-4}. Similar calculations are carried out for the following station ξ_{i+1}.

All the computations are performed in double-precision on the IBM 360/75 digital computer. The average computation time required for each station ξ_i ranges from 0·5 to 2 min depending on the value of ξ and N. The larger the value of ξ, the more the number of iterations required to satisfy the convergence criteria. The expansion coefficients $A(\nu_{0,\xi})$ and $A(\nu, \xi)$ appearing in equation (21) are evaluated with an iterative process using a 41-point Gaussian quadrature. The results of integration in the η-direction are obtained by using a Runge–Kutta scheme similar to the one developed by Nachtsheim and Seigert [9], which has the advantage of a built-in technique for correcting initial guesses. Defining error term based on the boundary conditions at the edge of the boundary layer as

$$\text{error} = f'^2 + (\theta - \theta_\infty)^2 + f''^2 + \theta'^2 \quad \text{(hot wall case)} \tag{34}$$

the correction for the initial guessed values of $f''(0)$ and $\theta'(0)$ is continued until the error term is less than 10^{-6}.

RESULTS

Figures 2 and 3 show the temperature and velocity profiles respectively at several axial

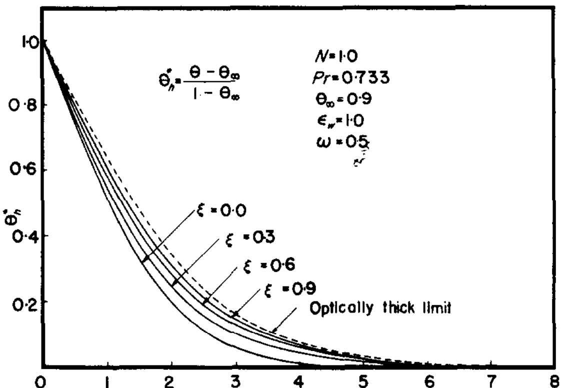

FIG. 2. Effect of the parameter ξ on the temperature profile.

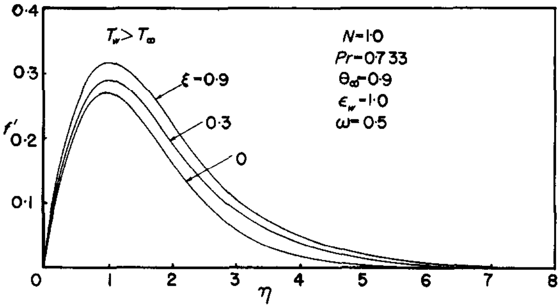

FIG. 3. Effect of the parameter ξ on the velocity profile.

positions between $\xi = 0$ and 0·9 for the case of hot wall. The temperature profile for $\xi = 0$ characterizes the non-radiating case and it agrees with that given by Ostrach [8]. As the value of ξ increases, the temperature profile approaches to that obtained by the optically thick limit approximation shown by the dotted line on the figure. In the case of the velocity distribution, increasing the value of ξ increases both the velocity and the velocity boundary layer thickness.

In Table 1 we present the numerical values

Table 1. Numerical results for velocity and temperature in the boundary layer ($\theta_\infty = 0·9$, $\varepsilon_w = 1·0$, $\xi = 0·1$, Pr = 0·733, $T_w > T_\infty$)

ω	N	η	f'	$(\theta - \theta_\infty)/(1 - \theta_\infty)$
0·0	0·1	0·0	0·0	1·0
		0·5	0·2322	0·7383
		1·0	0·2946	0·5197
		1·5	0·2689	0·3544
		2·0	0·2163	0·2429
		3·0	0·1300	0·1338
		4·0	0·0827	0·0925
		5·0	0·0531	0·0696
		6·0	0·0318	0·0517
		7·0	0·0163	0·0353
0·0	1·0	0·0	0·0	1·0
		0·5	0·2241	0·7472
		1·0	0·2755	0·5153
		1·5	0·2366	0·3285
		2·0	0·1707	0·1972
		3·0	0·0677	0·0653
		4·0	0·0217	0·0212
		5·0	0·0051	0·0055
0·5	0·1	0·0	0·0	1·0
		0·5	0·2292	0·7424
		1·0	0·2873	0·5178
		1·5	0·2566	0·3434
		2·0	0·1995	0·2242
		3·0	0·1083	0·1082
		4·0	0·0618	0·0677
		5·0	0·0358	0·0477
		6·0	0·0185	0·0323
0·5	1·0	0·0	0·0	1·0
		0·5	0·2237	0·7475
		1·0	0·2746	0·5148
		1·5	0·2350	0·3269
		2·0	0·1686	0·1945
		3·0	0·0650	0·0616
		4·0	0·0194	0·0178
		5·0	0·0039	0·0030
		Non-radiative case		
1·0		0·0	0·0	1·0
		0·5	0·2231	0·7475
		1·0	0·2731	0·5135
		1·5	0·2326	0·3240
		2·0	0·1653	0·1902
		3·0	0·0610	0·0560
		4·0	0·0164	0·0130
		5·0	0·0026	0·0004

Table 2. Numerical results for local Nusselt number ($\theta_\infty = 0.9$, $\varepsilon_w = 1.0$, $T_w > T_\infty$)

	ω	N	Pr	ξ	$Nu_x/(Gr_x/4)^{\frac{1}{4}}$
Effects of N	0·0	0·1	0·733	0·2	1·2618
		1			0·5755
		10			0·5143
		∞			0·5083
Effects of ω	0·0	0·1	0·733	0·1	0·7713
	0·5				0·6916
	0·9				0·6055
	1·0				0·5083
Effects of ξ	0·0	0·1	0·733	0	0·5083
				0·025	0·5594
				0·05	0·6120
				0·10	0·7713
				0·15	0·9951
				0·20	1·2618
Effects of Pr	0·0	0·1	0·733	0·1	0·7713
			10		1·3609
			100		2·3784

of velocity and temperature in the boundary layer as a function of η at $\xi = 0.1$ for different values of the parameters ω and N. The case $\omega = 1$ characterizes a purely scattering medium in which there is no interaction between radiation and convection, hence it represents a non-radiative case. An examination of this table reveals that the thickness of both velocity and temperature boundary layers increases with decreasing N and decreasing ω.

Table 2 shows the effects of the parameters N, ω, ξ and Pr on the local Nusselt number for a set of conditions specified on this table. The ratio $Nu_x/(Gr_x/4)^{\frac{1}{4}}$ increases with decreasing N, decreasing ω, increasing Prandtl number and increasing ξ. For large values of N or for $\omega = 1$ this ratio is the same as that for the non-radiative case.

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RAYONNEMENT AVEC CONVECTION LIBRE DANS UN MILIEU ABSORBANT, ÉMETTEUR ET DISPERSANT

Résumé—Le transfert thermique par rayonnement et convection naturelle simultanée depuis une plaque verticale immergée dans un fluide absorbant, émetteur, isotropiquement dispersant et gris est déterminé par la résolution des équations non similaires de quantité de mouvement et d'énergie. La part du problème relative au rayonnement est traitée exactement par l'application de la technique de développement en mode normal. On considère aussi bien le cas de la paroi chaude que celui de la paroi froide. Une analyse des paramètres est faite pour étudier les effets isolés des paramètres de l'albedo dispersant, de l'émissivité de la paroi, de l'épaisseur optique, de la conduction et du rayonnement sur la distribution de température dans la couche limite et sur le transfert thermique à la paroi. Les résultats de cette analyse peuvent être utilisés pour tester la précision des solutions obtenues par des méthodes approchées.

STRAHLUNG MIT FREIER KONVECTION IN EINEM ABSORBIERENDEN EMITTIERENDEN UND STREUENDEN MEDIUM

Zusammenfassung—Der Wärmeübergang durch gleichzeitige Strahlung und freie Konvektion von einer vertikalen Platte, die in ein absorbierendes, emittierendes, isotrop streuendes, graues Fluid eingetaucht ist, wird bestimmt durch Lösen der nicht ähnlichen Impuls- und Energiegleichungen. Der Strahlungsanteil des Problems wird durch die Anwendung der "normal-mode-expansion-technique" exakt beherrscht. Es wird sowohl der Fall der heissen als auch der kalten Wand betrachtet. Um die Einflüsse der einzelnen Streualbedos, des Emissionsvermögens der Wand, der optischen Dicke und des Leitungs-zu-Strahlungs-parameters auf die Temperaturverteilung in der Grenzschicht und den Wärmeübergang an der Wand zu studieren, wird eine Parameterübersicht durchgeführt. Die Ergebnisse dieser Analyse können dazu verwendet werden die Genauigkeit von Näherungslösungen zu überprüfen.

ИЗЛУЧЕНИЕ ПРИ СВОБОДНОЙ КОНВЕКЦИИ В ПОГЛОЩАЮЩЕЙ, ИЗЛУЧАЮЩЕЙ И РАССЕИВАЮЩЕЙ СРЕДЕ

Аннотация—Определяется теплоперенос при совместном излучении и свободной конвекции от вертикальной пластины, погруженной в поглощающую, излучающую, изотропно рассеивающую серую среду путем решения неавтомодельных уравнений количества движения и энергии. Часть задачи, относящаяся к лучистому переносу, решается точно с применением методики разложения по нормальным модам. Рассматриваются случаи как с холодной, так и горячей стенкой. Проводится обзор параметров с целью изучения влияния альбедо от единичного рассеивателя, эмиссионной способности стенки, оптической толщины, отношения теплопроводности к излучению на распространение температуры в пограничном слое и теплоперенос у стенки. Полученные результаты можно использовать для проверки точности решений, полученных с помощью приближенных методов.